Interplay of charge and spin in quantum dots: The Ising case

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(Received 15 July 2010; revised manuscript received 17 January 2011; published 3 August 2011)

The physics of quantum dots is depicted succinctly by the *universal Hamiltonian*, where only zero-mode interactions are included. In the case in which the latter involve charging and isotropic spin-exchange terms, this would lead to a non-Abelian action. Here we address an Ising spin-exchange interaction, which leads to an Abelian action. The analysis of this simplified yet nontrivial model shed light on a more general case of charge and spin entanglement. We present a calculation of the tunneling density of states and dynamic magnetic susceptibility. We explain how the latter can be used for an experimental determination of the exchange interaction strength.

DOI: 10.1103/PhysRevB.84.075307 PACS number(s): 73.23.Hk, 73.63.Kv, 75.30.Gw

I. INTRODUCTION

Significant progress in the study of the physics of quantum dots (QD's)[1] has been achieved following the introduction of the *universal Hamiltonian*[2,3] (UH). The latter facilitated the simplification of intricate electron-electron interactions within a QD in a controlled way. Within that scheme, interactions are represented as the sum of three spatially independent terms: charging, spin-exchange, and Cooper channel. Notably, even the inclusion of the first two terms turned out to be nontrivial: the resulting action is non-Abelian.[4]

To understand the complexity of such a problem, one can refer to the case of charging-only interaction. As was suggested by Kamenev and Gefen,[5] one can take the following steps in solving that problem: start from a fermionic action that includes an interaction term quadric in the (fermionic Grassman) variables, perform a Hubbard-Stratonovich transformation by introducing an auxiliary bosonic field, then perform a gauge transformation over the Grassman variables, and finally integrate them out. The resulting, purely bosonic, action is simple. In an imaginary time (Matsubara) picture, the action is quadratic in the bosonic components, which renders this action easily solvable. The trick of gauge-integrating over Grassman variables does not work for the non-Abelian case,[4] so an alternative approach is needed.

Attempts to account for charge and spin interactions in QD's have been reported earlier. Alhassid and Rupp[6,7] have found an exact solution for the partition function (and susceptibility); elements of their analysis were then incorporated in a master equation analysis of transport through the QD. More recently, an exact solution of the isotropic spin interaction model has been presented.[8] For the latter model, some quantities turn out to be particularly simple (e.g., the finite frequency spin susceptibility vanishes; evidently, there is no difference between longitudinal and transverse spin susceptibility). This means that the analysis of a model with anisotropy in the spin interaction is called for. A perturbation expansion in spin anisotropy has been reported earlier,[4] but it still remains desirable to consider an anisotropic model that can be analyzed exactly. By considering such a model, one would be able to understand the entanglement between charge and spin degrees of freedom, and also see in detail how a nonvanishing, complex spin susceptibility arises. This is the focal point of the present analysis.

In bulk systems, the exchange interaction competes with the kinetic energy leading to Stoner instability (SI).[9] In finite-size systems, a mesoscopic Stoner unstable regime may be a precursor of bulk thermodynamic SI. We consider here an Ising spin interaction. Such a model is Abelian, and complications due to noncommutativity of different terms in the action do not arise here. Also, such a model does not exhibit a mesoscopic Stoner unstable regime.[3] This means that at zero temperature, as the dimensionless parameter J/Δ (J being the exchange interaction strength and Δ the mean level spacing), the system abruptly switches from a paramagnetic to a (thermodynamic Stoner unstable) ferromagnetic phase. We stress that notwithstanding the simplicity of the model considered, spin-charge entanglement is present here, and nontrivial transverse ac susceptibility does arise. Some of our conclusions can be tested in principle in QD's made of materials close to the thermodynamic Stoner instability, e.g., Co impurities in a Pd or Pt host, Fe or Mn dissolved in various transition-metal alloys, Ni impurities in a Pd host, and Co in Fe grains, as well as new nearly ferromagnetic rare-earth materials.[10–12]

We consider the charge and spin response functions at temperatures $T \gg \Delta$ corresponding to QD's in the metallic regime, extensively studied experimentally (see, e.g., Refs. 1 and 13 and references therein). In this regime, many QD levels ($\mathcal{N} \sim T/\Delta$) are involved in quantum transport. Then the structure of single-particle states does not enter the calculations. In the opposite limit, $T \ll \Delta$, when only a *few* levels are involved in quantum transport, both elastic and inelastic cotunneling contribute to it in the Coulomb blockade valleys while sequential tunneling dominates at the peaks. This may lead to the spin and orbital *Kondo* effect[14] in nanoclusters [see also scanning tunneling microscopy (STM) experiment in Ref. 13 and theory in Ref. 15].

The outline of this paper is as follows. In Sec. II, we introduce our model Hamiltonian and the subsequent imaginary-time action. In Sec. III, we employ the technique of zero-dimensional functional bosonization,[4] which eventually allows us to express the single-particle Green's function as a product of the noninteracting Green's function and a term that depends on two bosonic fields. We then show how to reduce the problem to that of classical stochastic equations for the bosonic fields. Thus, in these sections we demonstrate the power of a

new method, *stochastic bosonization*, on a simple theoretical model, the physics of which is well understood. In Sec. IV, we express the grand-canonical partition function in terms of canonical ones, leading to both a mathematical and physical simplification of the calculation. In Sec. V, we calculate the tunneling density of states, and in Sec. VI we calculate longitudinal and transverse spin susceptibilities. Section VII presents a summary of the main results with some perspectives. We include some more technical calculations in four Appendixes.

II. HAMILTONIAN AND EFFECTIVE ACTION

We consider a normal-metal QD in the metallic regime, where the Thouless energy $E_{\rm Th}$ and the mean level spacing Δ satisfy $g \equiv E_{\rm Th}/\Delta \gg 1$ (g is the dimensionless conductance). In addition, the condition $E_c \gg \Delta$ is fulfilled in a relatively large QD as Δ scales with size L as $1/L^2$ while the charging energy $E_c \sim 1/L$. (Neglecting geometrical factors, for a 2D quantum dot $E_c \gg \Delta$ is equivalent to $L \gg a_B$, where a_B is the effective Bohr radius for electrons in the dot.) It is in this regime where the problem can be described in terms of the UH.

We restrict ourselves to a simplified version of the UH where the interaction in the Cooper channel is set to zero and the spin-exchange term is chosen to be a fully anisotropic Ising-like term, $-J\hat{S}_Z^2$, with a ferromagnetic exchange coupling, $J > 0$, where $\hat{S}_Z$ is the total spin of the dot in the $\hat{z}$ direction. This form of interaction is sufficient to bring about the Stoner instability phenomenon and other spin-related effects while avoiding calculational complexities inherent to a fully spin-symmetric model. Possible physical sources for such an anisotropy may include geometrical and/or molecular anisotropy, magnetic impurities in the system, or even the application of anisotropic mechanical pressure.

The complete form of the reduced UH is thus

$$H = \sum_{\alpha,\sigma} \varepsilon_\alpha a^\dagger_{\alpha,\sigma} a_{\alpha,\sigma} + E_c \left[\sum_{\alpha,\sigma} a^\dagger_{\alpha,\sigma} a_{\alpha,\sigma} - N_0 \right]^2 - \frac{J}{4} \left[\sum_{\alpha} a^\dagger_{\alpha,\sigma} \sigma^z_{\sigma\sigma'} a_{\alpha,\sigma'} \right]^2. \quad (1)$$

Here $\{\varepsilon_\alpha\}$ is a set of electronic levels in the dot, and N_0 in the charging term represents a positive background charge controlled via an external gate. We assume that the QD is either isolated or weakly coupled to the leads and in the Coulomb blockade regime. On the other hand, we will be considering the spin-disordered regime below the Stoner instability. So the parameters of the Hamiltonian (1) obey

$$J < \Delta \ll T \ll E_c, \quad (2)$$

where $T \equiv \beta^{-1}$ is the temperature.

The Euclidean action corresponding to the Hamiltonian (1) is given by

$$S[\overline{\Psi},\Psi] = \sum_\alpha \int_0^\beta d\tau \left\{ \overline{\Psi}_\alpha (\partial_\tau + \varepsilon_\alpha - \mu)\Psi_\alpha + E_c \left[\sum_\alpha \overline{\Psi}_\alpha \Psi_\alpha - N_0 \right]^2 - \frac{J}{4} \left[\sum_\alpha \overline{\Psi}_\alpha \sigma^z \Psi_\alpha \right]^2 \right\}, \quad (3)$$

where we use spinor notations $\overline{\Psi}_\alpha = (\overline{\psi}_{\uparrow\alpha}(\tau), \overline{\psi}_{\downarrow\alpha}(\tau))$. We introduce two auxiliary bosonic fields, $\varphi^c(\tau)$ and $\varphi^s(\tau)$, to decouple the Coulomb and exchange terms with the help of a standard Hubbard-Stratonovich (HS) transformation. This results in the following action:

$$S = S^c + S^s + S^{\rm mix}, \quad (4)$$

where

$$S^c = \int_0^\beta d\tau \left[\frac{\varphi^c(\tau)^2}{4E_c} - {\rm i}N_0\varphi^c(\tau) \right],$$
$$S^s = \int_0^\beta d\tau \frac{\varphi^s(\tau)^2}{J}, \quad (5)$$
$$S^{\rm mix} = \int_0^\beta d\tau \sum_\alpha \overline{\Psi}_\alpha [\partial_\tau + \varepsilon_\alpha - \mu + {\rm i}\varphi^c + \sigma^z\varphi^s]\Psi_\alpha.$$

Here σ^z is a Pauli matrix, the bosonic fields are periodic, and the fermionic fields are antiperiodic in τ with period β. This action is the starting point for all the subsequent calculations. We will use the functional bosonization approach as developed in Refs. 4 and 16: first we gauge out the mixed fermionic-bosonic terms in the action (5) and then integrate over the fermionic field, thus arriving at a purely bosonic action. After that, instead of dealing with this action directly, we will use a stochastic bosonization as described in the following section.

III. FROM FUNCTIONAL TO STOCHASTIC BOSONIZATION

In order to gauge out the mixed fermionic-bosonic terms in the action (5), we introduce a generalized gauge transformation, $\widetilde{\Psi}_\alpha = \mathcal{T}^{-1}\Psi_\alpha, \widetilde{\overline{\Psi}}_\alpha = \overline{\Psi}_\alpha\mathcal{T}$, with

$$\mathcal{T} = {\rm e}^{{\rm i}\theta^c(\tau)\mathcal{I} + \theta^s(\tau)\sigma^z} = \begin{pmatrix} {\rm e}^{{\rm i}\theta^c(\tau)+\theta^s(\tau)} & 0 \\ 0 & {\rm e}^{{\rm i}\theta^c(\tau)-\theta^s(\tau)} \end{pmatrix}.$$

"Gauging out" implies the following identity:

$$\overline{\Psi}_\alpha[\partial_\tau + {\rm i}\varphi^c(\tau) + \sigma^z\varphi^s(\tau)]\Psi_\alpha = \widetilde{\overline{\Psi}}_\alpha [\partial_\tau + \mathcal{A}]\,\widetilde{\Psi}_\alpha, \quad (6)$$

where $\mathcal{A}$ is some constant matrix. In order to fulfill (6), we require the gauge matrix $\mathcal{T}$ to obey

$$[\partial_\tau + {\rm i}\varphi^c(\tau) + \sigma^z\varphi^s(\tau)]\mathcal{T} = \mathcal{T}\mathcal{A}. \quad (7)$$

Since the bosonic fields are real, this equation separates into real and imaginary parts, corresponding to the exchange and charge channels. Using the substitution $\mathcal{A} = \mathcal{A}^s\sigma^z + {\rm i}\mathcal{A}^c$ for the constant matrix $\mathcal{A}$ in the matrix gauge equation (7), we have

$$\dot{\theta}^a(\tau) = \mathcal{A}^a - \varphi^a(\tau), \quad (8)$$

where a stands either for charge, c, or spin, s.

To determine the constants $\mathcal{A}^s$ and $\mathcal{A}^c$, we note that the antiperiodicity of the fermionic fields requires that $\mathcal{T}(\beta) = \mathcal{T}(0)$. This in turn implies $\theta^s(\beta) = \theta^s(0) + 2\pi{\rm i}n^s$ and $\theta^c(\beta) = \theta^c(0) + 2\pi N$ with integer n^s and N. Now we single out zero-Matsubara-frequency components of the bosonic fields $\varphi^a(\tau)$:

$$\varphi^a(\tau) = \varphi^a_0 + \widetilde{\varphi}^a(\tau), \quad \beta\varphi^a_0 \equiv \int_0^\beta d\tau\varphi^a(\tau). \quad (9)$$

Integrating Eq. (8) over τ from 0 to β results in $\mathcal{A}^c = \varphi_0^c + (2\pi/\beta)N$ and $\mathcal{A}^s = \varphi_0^s + (2\pi \mathrm{i}/\beta)n^s$, so that the gauge equation (8) reduce to the following form:

$$\dot{\theta}^c(\tau) = \frac{2\pi}{\beta}N - \widetilde{\varphi}^c(\tau), \tag{10a}$$

$$\dot{\theta}^s(\tau) = \frac{2\pi \mathrm{i}}{\beta}n^s - \widetilde{\varphi}^s(\tau). \tag{10b}$$

After the gauge transformation, the mixed action in Eq. (5) is reduced to the following quadratic fermionic action in terms of the transformed fields:

$$S_f = \int_0^\beta d\tau \sum_\alpha \overline{\widetilde{\Psi}}_\alpha(\tau)[\partial_\tau + \varepsilon_\alpha - \widetilde{\mu}_\sigma]\,\widetilde{\Psi}_\alpha(\tau). \tag{11}$$

The zeroth components of the bosonic fields (D5) enters Eq. (11) via the spin-dependent effective chemical potential μ_σ given by

$$\widetilde{\mu}_\sigma = \mu - \mathrm{i}\varphi_0^c - \sigma\varphi_0^s - \frac{2\pi \mathrm{i}}{\beta}(N + \sigma n^s), \tag{12}$$

where $\sigma = \pm 1$ for spin up and spin down, respectively.

The gauge equations (10) become important for correlation functions that are not gauge-invariant but depend on phase terms that are functions of the gauge parameters θ^c and θ^s (e.g., the Green's function calculated in the following section and Appendix A). These parameters are functionals of the bosonic fields $\widetilde{\varphi}^c(\tau)$ and $\widetilde{\varphi}^s(\tau)$, respectively. Thus, in order to calculate these phase terms, one should solve the gauge equations and then carry out the integration over the bosonic fields.[5,16]

Here, however, we consider an alternative method, which bypasses the need to carry out the functional integrals over $\widetilde{\varphi}^{c,s}(\tau)$. Even though in our case these integrations pose no great difficulty, the method we consider has general applicability and could be used in cases in which such integrations are impossible to perform analytically.

Our approach is to view the gauge equations (10) as classical Langevin equations governing the stochastic dynamics of θ^c and θ^s, with the bosonic fields playing the role of noise. The distribution of the noise is determined by the bosonic actions S^c and S^s, Eq. (5).

The Langevin equations can be mapped, via the standard tools of classical stochastic analysis,[17] to Fokker-Planck (FP) equations from which the time-dependent distribution functions for θ^c and θ^s can be determined. As an example, the form of the FP equation derived from Eq. (10a) is

$$\frac{\partial \mathcal{P}^c}{\partial \tau} = \left(\frac{2\pi}{\beta}N - \mathrm{i}\zeta\right)\frac{\partial \mathcal{P}^c}{\partial \theta} + E_c\frac{\partial^2 \mathcal{P}^c}{\partial \theta^2}, \tag{13}$$

where $\mathcal{P}^c$ is the distribution function for the gauge parameter θ^c, and ζ is a constant (details regarding the transition from Langevin to FP equations and their solution are given in Appendix C). Equation (13) is a standard diffusion equation with a drift term, the solution of which (with an appropriate initial condition) is simply a decaying Gaussian, explicitly given by Eq. (C5).

This distribution, and a similar one for θ_S, can now be used to calculate the averaging of any phase terms involving the gauge parameters in the calculation of non-gauge-invariant correlation functions. Thus we can, in effect, replace a functional integration with an integration over a finite number of parameters. This is an alternative method by which to integrate out the finite frequency components of the bosonic fields $\varphi^c(\tau)$ and $\varphi^s(\tau)$.

IV. SINGLE-PARTICLE GREEN'S FUNCTION: EFFECTIVE CHARGE QUANTIZATION

We begin with calculating the temperature Green's function (GF) in the grand-canonical ensemble, and we will show that in the Coulomb blockade regime, it reduces naturally to one in the canonical ensemble. Our starting expression is

$$\mathcal{G}_\sigma(\tau,\mu) = \sum_\alpha \mathcal{G}_{\alpha,\sigma}(\tau,\mu),$$
$$\mathcal{G}_{\alpha,\sigma} \equiv \frac{1}{\mathcal{Z}(\mu)}\int \mathcal{D}[\overline{\Psi}_\alpha\Psi_\alpha]\mathrm{e}^{-S[\overline{\Psi}_\alpha\Psi_\alpha]}\overline{\Psi}_{\alpha,\sigma}(\tau_i)\Psi_{\alpha,\sigma}(\tau_f), \tag{14}$$

where $\mathcal{G}_{\alpha,\sigma}$ is an auxiliary GF corresponding to a level ε_α, $S[\overline{\Psi}_\alpha\Psi_\alpha]$ is the α term in the Euclidean action (3), and $\tau \equiv \tau_f - \tau_i$.

After the HS transformation and gauge transform (6), the Gaussian integration over the quadratic fermionic action (11) is straightforward. The resulting GF of noninteracting electrons corresponding to this action, $\mathcal{G}^0_{\alpha,\sigma}(\tau,\widetilde{\mu}_\sigma)$, depends—via Eq. (12)—only on the zero-frequency component of the bosonic fields φ_0^a. This allows us to subdivide the remaining functional integration with the bosonic part of the action (5) into that over the zero-frequency, φ_0^a, and finite frequency, $\widetilde{\varphi}^a$, components, which results in the following expression:

$$\mathcal{G}_{\alpha,\sigma} = \Pi^c(\tau)\Pi^s(\tau)\frac{\langle\!\langle \mathcal{Z}^0(\widetilde{\mu})\mathcal{G}^0_{\alpha,\sigma}(\tau,\widetilde{\mu}_\sigma)\rangle\!\rangle_0}{\langle\!\langle \mathcal{Z}^0(\widetilde{\mu})\rangle\!\rangle_0}. \tag{15}$$

Here $\Pi^a(\tau)$ are the phase correlation functions resulting from the functional averaging of the charge or spin phase factors over the finite-frequency components of the appropriate fields, and $\langle\!\langle\cdots\rangle\!\rangle_0$ stand for the functional integrals over the zeroth-component fields φ_0^c and φ_0^s. All these functional integrals are defined in Eqs. (A1)–(A3) in Appendix A. Then $\mathcal{Z}^0(\widetilde{\mu}) = \mathcal{Z}^0_\uparrow(\widetilde{\mu}_\uparrow)\mathcal{Z}^0_\downarrow(\widetilde{\mu}_\downarrow)$ and $\mathcal{G}^0_{\alpha,\sigma}(\tau,\widetilde{\mu}_\sigma)$ is the grand-canonical partition function[18] of noninteracting electrons with the spin-dependent chemical potential $\widetilde{\mu}_\sigma$, defined by Eq. (12).

The charging effects can be fully accounted for by introducing winding numbers in the integration over φ_0^c:

$$\varphi_0^c = \omega_m + \frac{\widetilde{\varphi}_0^c}{\beta} \quad \omega_m = \frac{2\pi}{\beta}m, \tag{16}$$

where $-\pi < \widetilde{\varphi}_0^c \leqslant \pi$ and an integer m is a winding number. In the original work of Gefen and Kamenev,[5] these were not considered, leading to an incorrect final result. They were first introduced in the context of the charging interaction on small metallic grains by Efetov and Tschersich[19] within a Matsubara framework, and were finally implemented correctly by Sedlmayr, Yurkevich, and Lerner[20] within a Keldysh-technique framework. The introduction of the winding numbers (16) allows us to replace integration over φ_0^c with summation over all integers m and integration over $\widetilde{\varphi}_0^c$. The

sum over m is performed using the Poisson formula, which results in a new summation of the form

$$\sum_N \mathrm{e}^{-\beta E_c (N-N_0)^2} \times \mathcal{F}(N).$$

The Poisson resummation transforms summation over m into summation over the conjugate variable, N. In our case, φ_0^c represents a phase whose conjugate is evidently the particle number N. While the sum over the parameter m had many contributions [since $(\beta E_c)^{-1} \ll 1$], the sum over N contains, under the conditions (2), only two terms $N = N_0 \pm \frac{1}{2}$ near the Coulomb peak (N_0 is half an integer) and one term in the Coulomb valleys (i.e., everywhere outside of the region of width T near the peak): the contribution of all the other terms is exponentially suppressed. This is a manifestation of charge quantization in QD's.

In this way, we perform the integration in Eq. (15) to find (see Appendix A)

$$\mathcal{G}_{\alpha,\sigma}(\tau,\mu) = \frac{\widetilde{\Pi}^c(\tau)\widetilde{\Pi}^s(\tau)}{\widetilde{\mathcal{Z}}(\mu)} \sum_N \mathrm{e}^{-\beta E_c (N-N_0+\frac{\tau}{\beta})^2} \mathcal{I}_N, \tag{17}$$

$$\mathcal{I}_N \equiv \int_{-\infty}^{\infty} d\widetilde{\varphi}_0^s \mathrm{e}^{-\frac{[\widetilde{\varphi}_0^s]^2}{\beta J}} \int_{-\pi}^{\pi} \frac{d\widetilde{\varphi}_0^c}{2\pi} \mathrm{e}^{\mathrm{i}(N+\frac{\tau}{\beta})\widetilde{\varphi}_0^c} \mathcal{Z}^0(\widetilde{\mu}) \mathcal{G}^0_{\alpha,\sigma}(\tau,\widetilde{\mu}_\sigma), \tag{18}$$

where the reduced phase correlation functions $\widetilde{\Pi}^a$ are defined in Eq. (A4). The effective charge quantization in Eq. (17) makes it natural to change over from grand canonical to canonical quantities for a given N, followed by a weighted summation over N, where required. Let us stress that the canonical quantities are auxiliary, and we calculate in this way the grand-canonical GF of Eq. (17).

Expressing $\mathcal{I}_N$ via canonical quantities leads to an extra summation since $\mathcal{Z} = \sum_n \mathrm{e}^{\beta\mu n} Z_n$, etc. This calculation is detailed in Appendix B. The resulting full single-particle GF in imaginary time (following summation over all single-particle energy states) is given by

$$\mathcal{G}(\tau,\mu) = \frac{\pi T}{\Delta} \frac{\mathrm{e}^{-(E_c - J/4)|\tau|}}{\sin(\pi|\tau|T)} \frac{F(\tau)}{F(0)}, \tag{19}$$

where

$$F(\tau) = \sum_N \mathrm{e}^{-\beta E_c (\delta N)^2} \sum_{M=-N}^{N} \mathrm{e}^{-\frac{1}{4}\beta(\Delta - J)M^2 - \tau E_{N,M}},$$
$$\delta N \equiv N - N_0 - \frac{\mu}{2E_c}, \quad E_{N,M} \equiv 2E_c \delta N - \frac{JM}{2}. \tag{20}$$

The double summation above arises from replacing the grand-canonical partition function in terms of the sum over canonical ones, $\mathcal{Z}(\mu) = \sum_n \mathrm{e}^{\beta\mu n} Z_n$. The summation parameters are the electron number, N, and the total spin of the dot (in units of $\hbar/2$), M. Naturally, the GF is spin-independent: we are considering the regime of parameters, Eq. (2), below the Stoner instability where there is no symmetry breaking to distinguish opposite spin polarizations. Note that this result is valid in the regime (2), provided that

$$N\,\Delta \gg T, \quad (N - |M|)\,\Delta \gg T, \tag{21}$$

i.e., when the QD contains many electrons and is not very close to the Stoner instability. Moreover, under these conditions the sum over M in Eqs. (19) and (20) can be replaced by an integral from $-\infty$ to $+\infty$, and the exponent of $\frac{J^2\tau^2}{4\beta(\Delta - J)}$ resulting from this integration can be totally neglected. With the same accuracy, we should neglect the exchange energy J in the exponent in Eq. (19) disregarding a renormalization (shift) of the charging energy E_c due to spin-spin interaction (cf. Ref. 4). Thus we find

$$\mathcal{G}(\tau,\mu) = \frac{\pi \mathrm{e}^{-E_c|\tau|}}{\beta\Delta \sin\left(\frac{\pi|\tau|}{\beta}\right)} \frac{1}{\widetilde{\mathcal{Z}}} \sum_N \mathrm{e}^{-E_c[\beta(\delta N)^2 - 2\tau\delta N]}, \tag{22}$$

so that under conditions (2) and (21)—not surprisingly—the one-particle GF is independent of the exchange part of the universal Hamiltonian (1). Such a dependence would emerge only very close to the Stoner instability, when $|\Delta - J|/J \ll 1$, but this parametric region is beyond the scope of the presented technique. While Eq. (22) provides a representation of the single-particle GF in the limit (2), the full dependence on J and Δ is still preserved in the exact expressions (17) and (18). However, the limit $J \sim \Delta \sim T$ cannot be described within the present formalism and requires separate analysis.

V. TUNNELING DENSITY OF STATES

The tunneling density of states (TDoS), $\nu(\varepsilon)$, can be directly related to the conductance of the QD in the limit of weak coupling to the leads and is thus a quantity of great importance. The TDoS is given by $\nu(\varepsilon) = -\frac{1}{\pi} \mathrm{Im}\, \mathcal{G}^R(\varepsilon)$, where the retarded GF, $G^R(\varepsilon)$, is a Fourier transform of the GF in real time, $G(t,\mu)$, obtained from Eq. (22) by the straightforward analytical continuation from the upper half-plane. Since $\mathcal{G}(\tau,\mu)$ is independent of the exchange energy under the conditions (2) and Eq. (21), so is the TDoS.[20]

For tutorial purposes, we use the results of Appendixes A and B to derive a more general expression for $\nu(\varepsilon)$, valid for any relation between the parameters in Eqs. (2) and (21), and we show how it goes over to the known expression[20] under conditions (2) and (21).

Using the GF in the ε representation, Eq. (B7), and performing the summation over all the levels as described at the end of Appendix B, we find

$$\frac{\nu(\varepsilon)}{\nu_0} = \frac{1}{\widetilde{\mathcal{Z}}} \sum_N \sum_{M=-N}^{N} \mathrm{e}^{-\beta E_c (N - \widetilde{N}_0)^2 - \frac{1}{4}\beta(\Delta - J)M^2} \times [1 - n(\varepsilon - \bar{\mu} - \xi_{N,M}) + n(\varepsilon - \bar{\mu} - \xi_{N-1,M-1})], \tag{23}$$

where we have defined

$$\xi_{N,M} \equiv 2E_c\left(N - \widetilde{N}_0 + \tfrac{1}{2}\right) - \tfrac{1}{2}J\left(M + \tfrac{1}{2}\right) \tag{24}$$

and

$$\bar{\mu} \equiv \tfrac{1}{2}\Delta(N + M), \quad \widetilde{N}_0 \equiv N_0 + \frac{\mu}{2E_c}, \tag{25}$$

while $\nu_0 = 2/\Delta$ is the TDoS in the absence of interactions, $n(\epsilon) \equiv [1 + \mathrm{e}^{\beta\epsilon}]^{-1}$.

Equation (23) is the general expression for the TDoS for any combination of parameters for a many-electron dot. When the inequalities (2) and (21) are satisfied, we can easily sum over M as described at the end of the previous section and then limit

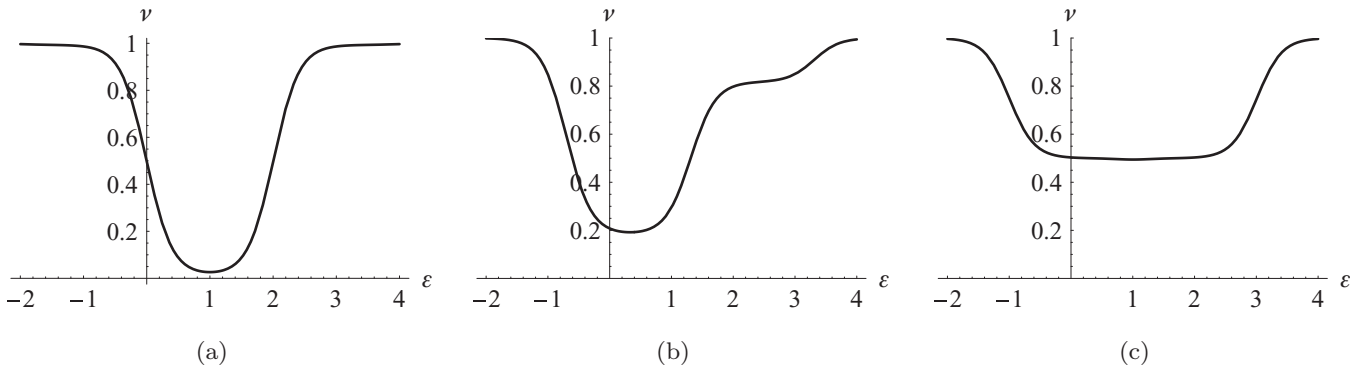

FIG. 1. TDoS (in units of ν_0) as a function of $\epsilon \equiv \varepsilon/E_c$ for $T = 0.2E_c$ and $\Delta/T = 0.1$ in (a) a CB valley ($\widetilde{N}_0 = 100$), (b) an intermediate region ($\widetilde{N}_0 = 100.35$), and (c) a CB peak ($\widetilde{N}_0 = 100.5$).

the summation over N to the two terms for which the value of $|N - \mathcal{N}_0|$ is minimal (although deep in the Coulomb valley, only one term is actually contributing). The resulting TDoS is independent of J (or, more precisely, tiny J-dependent corrections are beyond the accuracy of current calculations and thus omitted) and coincides with that obtained in Ref. 20:

$$\frac{\nu(\varepsilon)}{\nu_0} = \frac{U(\varepsilon - \xi_N) + e^{-\beta(\xi_N - \bar{\mu})} U(\varepsilon - \xi_{N+1})}{1 + e^{-\beta(\xi_N - \bar{\mu})}}, \tag{26}$$

where $U(\varepsilon - \xi_N) \equiv n(\varepsilon - \xi_{N-1} - \bar{\mu}) + 1 - n(\varepsilon - \xi_N - \bar{\mu})$, and ξ_N is obtained from $\xi_{N,M}$ by setting $J = 0$ in Eq. (24). We illustrate the dependence of ν on energy for integer, half-integer, and intermediate values of $\widetilde{N}_0$ in Fig. 1, for a specific choice of parameter values T and Δ. Its dependence on temperature at the bottom of a Coulomb blockade valley is depicted in Fig. 2. It is important to note that the TDoS obtained in the Coulomb valleys is not physical since we neglect cotunneling contributions; however, the T dependence near the peak will be obtained as a linear combinations of those shown in Fig. 2.

Note that for any given set of parameters, the center of the TDoS curve is at $\varepsilon_0 = \frac{1}{2}\Delta\widetilde{N}_0 - 2E_c(N - \widetilde{N}_0)$ and thus a function of $\widetilde{N}_0$, Eq. (25). With this moving from one Coulomb valley to the next, the TDoS curve is shifted by $\Delta/2$ due to adding an extra electron to the dot, which raises the effective chemical potential and thus shifts the TDoS curve. That is the reason for the "half-gap" in the TDoS at the degeneracy point.

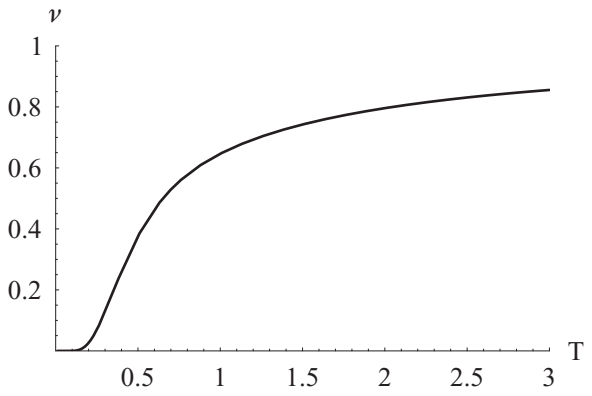

FIG. 2. Dependence of the TDoS (in units of ν_0) on the temperature (measured in E_c) at the bottom of a CB valley ($\widetilde{N}_0 = 70$) for $\Delta/E_c = 0.02$.

VI. MAGNETIC SUSCEPTIBILITY

We now turn to calculating the longitudinal and transverse magnetic susceptibilities of the system.

It is clear that only the static component of the longitudinal susceptibility is nonzero due to the lack of spin-flip processes in the Ising model.[21] A direct calculation of the correlation function $\langle S_z(\tau)S_z(0)\rangle$ shows this to be τ-independent, as expected. The static susceptibility is given by

$$\chi_{zz} = \frac{1}{\beta}\lim_{h\to 0}\frac{d^2}{dh^2}\ln \mathcal{Z}(h), \tag{27}$$

where $\mathcal{Z}(h)$ is the partition function of the system calculated in the presence of the following source term in the action:

$$S_h = -\frac{h}{2}\int_0^\beta d\tau \sum_\alpha \overline{\Psi}_\alpha \sigma^z \Psi_\alpha. \tag{28}$$

The calculation is straightforward, leading to the result

$$\mathcal{Z}(h) = \kappa \exp\left\{\frac{\beta^2 h^2}{4\beta(\Delta - J)}\right\},$$

with κ being some irrelevant constant. Plugging this into the definition (27) yields the well-known expression

$$\chi_{zz}(\omega = 0) = \frac{1}{2}\frac{1}{\Delta - J}. \tag{29}$$

As expected, the static susceptibility is independent of the number of particles on the dot, external gate voltage, charging effects, etc.

We now turn to a calculation of the transverse magnetic susceptibility. This quantity is inherently different from the longitudinal one since it is dynamic: the model allows for transitions between different transverse spin-polarization states.

We define the dynamic transverse susceptibility in imaginary time as

$$\frac{1}{\beta}\chi^{+-}(\tau) = \langle \sigma^+(0)\sigma^-(\tau)\rangle, \tag{30}$$

where $\sigma^+ = \sum_\alpha \overline{\Psi}_{\alpha\uparrow}\Psi_{\alpha\downarrow}$ and $\sigma^- = \sum_\alpha \overline{\Psi}_{\alpha\downarrow}\Psi_{\alpha\uparrow}$. Thus we need to calculate the functional average of

$$\sum_{\alpha,\beta} \overline{\Psi}_{\alpha\uparrow}(0)\Psi_{\alpha\downarrow}(0)\overline{\Psi}_{\beta\downarrow}(\tau)\Psi_{\beta\uparrow}(\tau),$$

with the action given by Eq. (3). The procedure closely follows that of the calculation of the GF described in Sec. IV. The final outcome of this calculation is

$$\chi^{+-}(\tau) = \frac{\beta e^{J\tau}}{\widetilde{\mathcal{Z}}(\mu)} \sum_N e^{-\beta E_c (N-\widetilde{N}_0)^2} \sum_{M=-N}^{N} \left\{ e^{-\frac{1}{4}\beta(\Delta-J)M^2} \times e^{J\tau M} \sum_\alpha [1 - n_\alpha(\bar{\mu}_\uparrow)] n_\alpha(\bar{\mu}_\downarrow) \right\}, \quad (31)$$

where $\bar{\mu}_\sigma \equiv N_\sigma \Delta$ and N_σ is the total number of electrons with the spin projection $\sigma = \uparrow, \downarrow$.

Fourier-transforming the result of Eq. (31) to Matsubara frequencies and then performing a simple analytic continuation, we find the imaginary part of the physical response function $\chi^{+-}(\omega)$:

$$\mathrm{Im}\chi^{+-}(\omega) = \frac{\sqrt{\pi\beta(\Delta - J)}}{2J} e^{\frac{\beta}{4}[(\Delta+J)-(\Delta-J)\frac{\omega^2}{J^2}]} \times \left(1 + \frac{\omega}{J}\right) \frac{\sinh\left[\frac{\beta\omega}{2}\right]}{\sinh\left[\frac{\beta\Delta}{2}\left(1+\frac{\omega}{J}\right)\right]}. \quad (32)$$

This function is depicted in Fig. 3. The most salient features are a linear dependence at the origin and the existence of a peak at a certain ω_0. Both the slope at the origin and the value of ω_0 can be used used to characterize an experimentally obtained curve of the transverse magnetic susceptibility as a function of frequency. We find the slope at $\omega \to 0$ as

$$\frac{1}{\beta}\mathrm{Im}\chi^{+-}(\omega \to 0) \approx \frac{\omega}{2J}\sqrt{\frac{\pi}{\beta\Delta}}, \quad (33)$$

where the approximation was made consistent with the inequality (2). Under the same condition, the peak frequency is given by

$$\omega_0 \approx \sqrt{\frac{2J^2}{\beta(\Delta - J)}}. \quad (34)$$

Yet another parameter of interest is the full width at half-maximum (FWHM). Numerical analysis shows that it is proportional to the resonance frequency: $W \approx 1.59\omega_0$. This result was derived by numerically obtaining the FWHM for various values of ω_0 and fitting the results to a linear curve, as shown in Fig. 4.

The imaginary part of the susceptibility represents the systems capacity to absorb and dissipate magnetic energy at a nonzero frequency. For the static susceptibility, only a real part is finite. A simple calculation leads to

$$\mathrm{Re}\chi^{+-}(\omega = 0) = \frac{1}{\Delta} e^{\frac{\beta}{4}(\Delta+J)} \approx \frac{1}{\Delta}. \quad (35)$$

Note that in the limit $J = 0$, we recover the well-known identity $\chi^{+-} = 2\chi_{zz}$ for the static susceptibilities. The real part of χ^{+-} at finite frequencies can be found either directly or via the Kramers-Kronig relations, but we do not present the result here as it has little physical relevance.

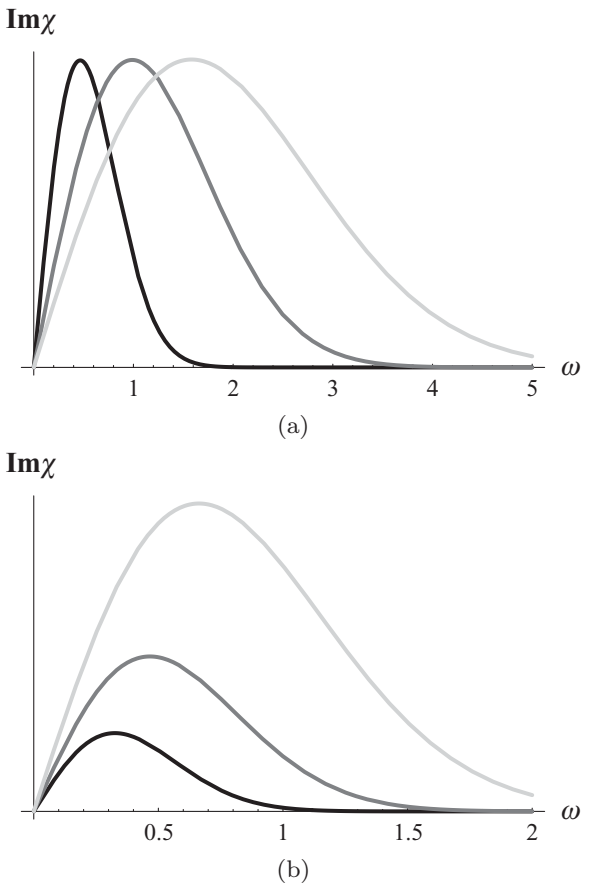

FIG. 3. $\frac{1}{\beta}\mathrm{Im}\chi^{+-}$ as a function of frequency ω (in units of Δ) for (a) $\frac{\Delta}{T} = 0.1$ and $\frac{J}{\Delta} = 0.1, 0.2$, and 0.3 for the left, center, and rightmost curves, respectively, and (b) $\frac{J}{\Delta} = 0.1$ and $\frac{\Delta}{T} = 0.05, 0.1$, and 0.2 for the top, center, and bottom curves, respectively.

As in the case of the longitudinal magnetic susceptibility, it is clear that the transverse susceptibility is not affected by the charging interaction in the dot. Once again we see that under conditions (2) and (21), the charge and spin degrees of freedom are effectively decoupled.

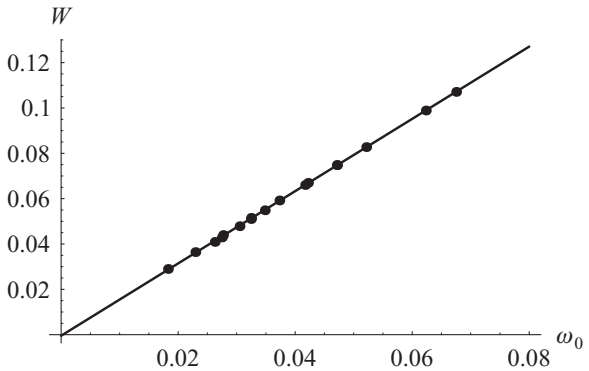

FIG. 4. Fit of numerically acquired data for FWHM to function $W = \alpha\omega_0$, yielding $\alpha = 1.59$. R^2 for this fit is 0.999.

VII. SUMMARY

The main results of this work fall into three basic categories: the single-particle GF, the TDoS, and the magnetic susceptibilities. The results for all three classes of correlation functions were obtained by means of the functional bosonization approach combined with the solution of classical stochastic equations for the bosonic fields. We have considered the Ising version of the *universal Hamiltonian* for describing the interplay of the spin and charge degrees of freedom in metallic quantum dots. Such a model is Abelian and therefore does not include the physics of noncommutative variables. It also does not exhibit the mesoscopic Stoner instability regime. Nevertheless, the spin-charge entanglement is present, being manifested in, e.g., nontrivial ac spin susceptibility. The model, being a simplified version of the quantum universal Hamiltonian model, gives a qualitatively correct description of the thermodynamics and transport through nanostructures in the vicinity of the *thermodynamic* Stoner instability point. The stochastic bosonization appears to be a very powerful tool for the treatment of Abelian gauge theories and a promising method for solving non-Abelian models corresponding to isotropic and anisotropic quantum limits of the universal Hamiltonian. The theory of thermodynamic Stoner instability and its influence on transport through a single-electron transistor can be tested experimentally in quantum dot device granular systems[10–12] and nanoclusters.[13,15] While the partition function and TDoS can be calculated using considerably simpler methods, the general equations for the GF and dynamical magnetic susceptibility require powerful machinery of either the functional or stochastic bosonization.

We summarize below the central results and key observations reported in the paper.

(i) *Canonical variables and charge quantization*. In our calculation of the GF, the tools we used and the choices made not only allowed us to carry out a nonperturbative calculation, but also had physical significance. The use of functional bosonization and generalized gauge transformations and the implementation of winding numbers, as well as the transformation to conjugated variables via the Poisson resummation, led us to employ canonical quantities. The latter is a consequence of strong charging interaction.

(ii) *Regimes of validity.* The transition to canonical quantities, namely the introduction of the canonical partition function, also led to further insight with regard to the various physical regimes in which the system may be found. Our calculation of the canonical partition function itself [and the associated quantity $\mathcal{Z}_N(\phi_\alpha)$] imposed limitations on the physical parameters involved. We found that the system must be large enough (meaning a large number of electrons) and far below the Stoner instability point. We had to self-consistently assume that the fluctuations in the systems magnetization were much smaller than the system size. This corresponds to a requirement that the system be far from a phase-transition point, which in our case is the SI point.

(iii) *Spin-charge entanglement*. The introduction of the canonical partition functions led directly to a summation over all possible values of the magnetization. These are of course limited to $|M| < N$. Since the number of particles itself is controlled by the charging interaction when in the CB regime, and the fluctuations of the magnetization are influenced by the exchange interaction, this can be seen as a form of coupling between the charge and spin degrees of freedom. The coupling between the two interaction channels becomes important as the magnitude of magnetization fluctuations increases, i.e., as one approaches the SI point. Only then do values of M that approach the system size become accessible and, consequently, of physical importance. Far below the SI point, the spin-charge coupling is very weak, and the effects of interplay are minimal. Our calculation of the TDoS showed the exchange interaction to have an extremely negligible effect. The magnetic susceptibilities in turn showed no dependence on the charging interaction.

(iv) *Determining J and Δ*. The calculation of the transverse magnetic susceptibility is, to our knowledge, a new result, and perhaps the most important in this work. As we have discussed previously, the importance of this result is that it provides an experimental method to determine the values of the parameters J and Δ. Our result is a direct prediction of the absorption spectrum of the system, and as such should be amenable to experimental measurement. The various curve characteristics that we derived, including the slope at $\omega \to 0$, the location of the resonance frequency, and the FWHM, should, in principle, through their dependence on J and Δ, allow these values to be ascertained from such a measurement.

ACKNOWLEDGMENTS

We acknowledge useful interaction with I. Burmistrov, I. Yurkevich, and O. Zilberberg. In particular, we thank Zeev Schuss for illuminating discussions concerning stochastic quantization. This work has been supported by SPP 1285 "Spintronics," Minerva Foundation, German-Israel GIF, Israel Science Foundation, EU project GEOMDISS, and the EPSRC grant T23725/01. M.K. acknowledges the support of the Einstein Minerva Center during his visits to WIS.

APPENDIX A: GRAND-CANONICAL SINGLE-PARTICLE GREEN'S FUNCTION

In this appendix, we present a detailed nonperturbative calculation of the single-particle Green's function (GF) for our model system (1). The GF itself was used to derive the tunneling density of states (TDoS), but its calculation also serves to show the methodology used in calculating the various other quantities considered in this work.

As discussed in Sec. II, a Hubbard-Stratonovich (HS) transformation is applied, reducing the action to the form presented in Eqs. (4) and (5). Carrying out the Gaussian integration over the fermionic fields after the gauge transformation (10), we obtain the GF as follows:

$$\mathcal{G}_{\alpha,\sigma}(\tau,\mu) = \Pi^c(\tau)\Pi^s(\tau) \times \frac{\int_{-\infty}^{\infty} d\varphi_0^c e^{-S_0^c[\varphi_0^c]} \int_{-\infty}^{\infty} d\varphi_0^s e^{-S_0^s[\varphi_0^s]} \left[\mathcal{Z}^0(\widetilde{\mu})\mathcal{G}^0_{\alpha,\sigma}(\tau,\widetilde{\mu}_\sigma)\right]}{\int_{-\infty}^{\infty} d\varphi_0^c e^{-S_0^c[\varphi_0^c]} \int_{-\infty}^{\infty} d\varphi_0^s e^{-S_0^s[\varphi_0^s]} \mathcal{Z}^0(\widetilde{\mu})}. \tag{A1}$$

Here $\mathcal{Z}^0(\widetilde{\mu}) = \mathcal{Z}^0_\uparrow(\widetilde{\mu}_\uparrow)\mathcal{Z}^0_\downarrow(\widetilde{\mu}_\downarrow)$ and $\mathcal{G}^0_{\alpha,\sigma}(\tau,\widetilde{\mu}_\sigma)$ are the grand-canonical partition function[18] and GF of noninteracting electrons with the spin-dependent chemical potential $\widetilde{\mu}_\sigma$,

defined by Eq. (12). Both $\mathcal{Z}^0$ and $\mathcal{G}^0$ are functions of the zero-Matsubara components φ_0^c and φ_0^s of the bosonic fields, over which the integration in Eq. (A1) is carried out with

$$S_0^c = \frac{\beta[\varphi_0^c]^2}{4E_c} - \mathrm{i}\beta N_0\varphi_0^c, \quad S_0^s = \frac{\beta[\varphi_0^s]^2}{J}. \tag{A2}$$

The functional integration over the remaining components of the bosonic fields results in the appearance of the phase-correlation functions:

$$\begin{aligned}\Pi^c(\tau) &= \langle \mathrm{e}^{\mathrm{i}[\theta^c(\tau_f)-\theta^c(\tau_i)]}\rangle_{\widetilde{\varphi}^c},\\ \Pi^s(\tau) &= \langle \mathrm{e}^{\sigma[\theta^s(\tau_f)-\theta^s(\tau_i)]}\rangle_{\widetilde{\varphi}^s}.\end{aligned} \tag{A3}$$

The functional averaging above is carried out with the weights $\exp[-\widetilde{S}^{c,s}]$, where $\widetilde{S}^{c,s}$ are obtained from the appropriate bosonic action in Eq. (5) by subtracting the zeroth Matsubara components of Eq. (A2).

The calculation of the correlation functions of Eq. (A3) is carried out in Appendix C using the tools of stochastic analysis. The results are

$$\begin{aligned}\Pi^c(\tau) &= \mathrm{e}^{-E_c(|\tau|-\frac{\tau^2}{\beta})}\mathrm{e}^{2\pi\mathrm{i}N\frac{\tau}{\beta}} \equiv \widetilde{\Pi}^c(\tau)\mathrm{e}^{2\pi\mathrm{i}N\frac{\tau}{\beta}},\\ \Pi^s(\tau) &= \mathrm{e}^{\frac{J}{4}(|\tau|-\frac{\tau^2}{\beta})}\mathrm{e}^{2\pi\mathrm{i}n^s\frac{\tau}{\beta}\sigma^z} \equiv \widetilde{\Pi}^s(\tau)\mathrm{e}^{2\pi\mathrm{i}N\frac{\tau}{\beta}}.\end{aligned} \tag{A4}$$

At this point, we introduce the winding numbers, as discussed in Sec. IV of the main text. Following the transition $\varphi_0^c = \omega_m + \frac{\widetilde{\varphi}_0^c}{\beta}$, and utilizing the identities $\mathcal{Z}^0(\mu - i\omega_m) = \mathcal{Z}^0(\mu)$ and $\mathcal{G}^0(\tau,\mu - i\omega_m) = \mathrm{e}^{-i\omega_m\tau}\mathcal{G}^0(\tau,\mu)$, we end up with

$$\mathcal{G}_{\alpha,\sigma}(\tau,\mu) = \frac{\widetilde{\Pi}^c(\tau)\widetilde{\Pi}^s(\tau)}{\widetilde{\mathcal{Z}}(\mu)}\sum_m \mathrm{e}^{2\pi\mathrm{i}(N_0-\frac{\tau}{\beta})m-\frac{\pi^2m^2}{\beta E_c}}\mathcal{I}_m,$$

$$\mathcal{I}_m \equiv \int_{-\infty}^{\infty} d\widetilde{\varphi}_0^s \int_{-\pi}^{\pi} d\widetilde{\varphi}_0^c \mathrm{e}^{-\frac{[\widetilde{\varphi}_0^s]^2}{\beta J}-\frac{[\widetilde{\varphi}_0^c]^2}{4\beta E_c}+\widetilde{\varphi}_0^c(\mathrm{i}N_0-\frac{\pi m}{\beta E_c})}\mathcal{Z}^0(\widetilde{\mu})\mathcal{G}^0_{\alpha,\sigma}.$$

The grand partition function $\widetilde{\mathcal{Z}}(\mu)$ above is represented by the same double-integral and sum with $\mathcal{G}^0$ replaced by 1. The exponential factors involving N and n^s arising from the phase-correlation functions and the noninteracting GF cancel each other out exactly. This is hardly surprising as they are completely arbitrary.

The summation over the winding numbers above can be performed using the Poisson formula

$$\sum_{k=-\infty}^{\infty} f(2\pi k) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}mx}f(x)\,dx. \tag{A5}$$

This results in the expression for the GF given by Eqs. (17) and (18) in the main text.

APPENDIX B: CALCULATIONS IN AN AUXILIARY CANONICAL ENSEMBLE

We express $\mathcal{Z}^0(\mu)$ in Eqs. (17) and (18) via the sum of the canonical partition functions for a system of n noninteracting electrons, Z_n^0, using the standard relation

$$\mathcal{Z}^0(\mu) = \prod_\alpha[1+\mathrm{e}^{-\beta(\varepsilon_\alpha-\mu)}] = \sum_n \mathrm{e}^{\beta\mu n}Z_n^0. \tag{B1}$$

To express the results of further integration in a convenient way, we also define the grand-canonical and canonical partition functions with one level, ε_α, excluded,

$$\mathcal{Z}^0(\not{\alpha},\mu) = \prod_{\alpha'\neq\alpha}[1+\mathrm{e}^{-\beta(\varepsilon'_\alpha-\mu)}] = \sum_n \mathrm{e}^{\beta\mu n}Z_n^0(\not{\alpha}). \tag{B2}$$

Then we substitute into Eq. (18) the finite-temperature GF of noninteracting fermions,

$$\mathcal{G}^0_{\alpha,\sigma}(\tau>0,\mu) = \mathrm{e}^{-(\varepsilon_\alpha-\mu)\tau}[1-n_{\alpha,\sigma}(\mu)], \tag{B3}$$

where $n_{\alpha,\sigma}(\mu)$ is the Fermi-Dirac occupation factor. We limit the calculation to $\mathcal{G}(\tau>0)$ since $\mathcal{G}(\tau) = -\mathcal{G}(\tau+\beta)$. Recalling that $\mathcal{Z}^0(\widetilde{\mu}) = \mathcal{Z}^0_\uparrow(\widetilde{\mu}_\uparrow)\mathcal{Z}^0_\downarrow(\widetilde{\mu}_\downarrow)$, we cast Eq. (18) into the form

$$\begin{aligned}\mathcal{I}_N = &\int_{-\infty}^{\infty} d\widetilde{\varphi}_0^s \mathrm{e}^{-\frac{[\widetilde{\varphi}_0^s]^2}{\beta J}}\int_{-\pi}^{\pi}\frac{d\widetilde{\varphi}_0^c}{2\pi}\mathrm{e}^{\mathrm{i}N\widetilde{\varphi}_0^c}\mathrm{e}^{-(\varepsilon_\alpha-\mu+\sigma\frac{\widetilde{\varphi}_0^s}{\beta})\tau}\\ &\times\sum_{m,n}\mathrm{e}^{[\beta\mu(m+n)-\mathrm{i}\widetilde{\varphi}_0^c(m+n)-\sigma(m-n)\widetilde{\varphi}_0^s]}\mathcal{Z}_{\sigma,m}(\not{\alpha})\mathcal{Z}_{-\sigma,n}.\end{aligned}$$

Carrying out the integration over $\widetilde{\varphi}_0^c$ yields a Krönecker delta $\delta_{N,n+m}$. Performing the Gaussian integration over $\widetilde{\varphi}_0^s$ and defining $M = m-n$, we find

$$\mathcal{I}_N = \sum_{M=-N}^{N}\mathrm{e}^{\beta\mu N+\frac{1}{4}\beta J(M+\tau/\beta)^2-(\varepsilon_\alpha-\mu)\tau}Z_{\frac{N+M}{2}}(\not{\alpha})Z_{\frac{N-M}{2}}.$$

Substituting this into Eq. (17) yields, after straightforward algebraic manipulations,

$$\begin{aligned}\mathcal{G}_{\alpha,\sigma}(\tau>0,\mu) = &\frac{1}{\widetilde{\mathcal{Z}}(\mu)}\sum_N\sum_{M=-N}^{N}Z_{\frac{N+M}{2}}(\not{\alpha})Z_{\frac{N-M}{2}}\\ &\times\mathrm{e}^{-\beta E_c(N-N_0)^2+\beta\mu N+\frac{1}{4}\beta JM^2-(\varepsilon_\alpha+\xi_{N,M})\tau},\end{aligned} \tag{B4}$$

where $\xi_{N,M}$ are defined in Eq. (24).

The canonical partition functions Z_N and $Z_N(\not{\alpha})$ are evaluated in Appendix D, resulting in

$$Z_N = \mathrm{e}^{-\frac{1}{2}\beta\Delta N^2}, \quad Z_N(\not{\alpha}) = [1-n_\alpha(\bar{\mu}_0)]\,Z_N, \tag{B5}$$

where the Fermi factor for the αth level, $n_\alpha(\bar{\mu}_0) \equiv [1+\mathrm{e}^{\beta(\varepsilon_\alpha-\bar{\mu}_0)}]^{-1}$, is taken with the auxiliary chemical potential $\bar{\mu}_0 \equiv N\Delta$.

Substituting Eq. (B5) into Eq. (B4), we find

$$\begin{aligned}\mathcal{G}_\alpha(\tau>0,\mu) = &\frac{1}{\widetilde{Z}}\sum_N\mathrm{e}^{-\beta E_c(N-\widetilde{N}_0)^2}\\ &\times\sum_{M=-N}^{N}\mathrm{e}^{-\frac{1}{4}\beta(\Delta-J)M^2}[1-n_\alpha(\bar{\mu})]\mathrm{e}^{-\xi_\alpha(N,M)\tau},\end{aligned} \tag{B6}$$

where $\bar{\mu}$ and $\widetilde{N}_0$ are defined in Eq. (25) and the auxiliary partition function $\widetilde{Z} \equiv F(0)$ is given by Eq. (20) in the main text. The result is naturally spin-independent. Technically, the formal spin dependence vanished when calculating the integral $\mathcal{I}_N$, Eq. (18). The GF for negative τ can be obtained from Eq. (B6) using $\mathcal{G}(-\tau) = -\mathcal{G}(\beta-\tau)$.

Now we find the full GF by summing over all single-particle states ε_a. This summation is carried out in the usual way by making the substitution $\sum_\alpha \mathcal{G}_\alpha \to \Delta^{-1}\int_0^\infty \mathcal{G}(\varepsilon_\alpha)d\varepsilon_\alpha$, i.e., effectively by averaging over disorder by introducing the mean level spacing Δ. This leads to Eq. (19) in the main text.

Finally, we write the GF in the energy representation. Making the standard analytical continuation to the real time, $\tau \to it$, and Fourier transforming the GF to the energy domain, we obtain the retarded GF used in the calculation of the TDoS as follows:

$$G^R_\alpha(\varepsilon) = \frac{1}{\widetilde{Z}}\sum_N \sum_{M=-N}^{N} \mathrm{e}^{-\beta E_c(N-\widetilde{N}_0)^2-\frac{1}{4}\beta(\Delta-J)M^2} \times\left[\frac{1-n_\alpha(\bar{\mu})}{\varepsilon-\varepsilon_\alpha-\xi_{N,M}+\mathrm{i}0}+\frac{n_\alpha(\bar{\mu})}{\varepsilon-\varepsilon_\alpha-\xi_{N-1,M-1}+\mathrm{i}0}\right]. \tag{B7}$$

APPENDIX C: PHASE-CORRELATION FUNCTIONS AND STOCHASTIC ANALYSIS

Here we use stochastic analysis to calculate the phase-correlation function Π^c defined in Eq. (A3). Π^s has been calculated in exactly the same manner.

We note that the gauge equation (10a) can be viewed as a Langevin equation wherein the field $\widetilde{\varphi}^c(\tau)$ plays the role of the stochastic force (noise), the distribution of which is governed by the action $S_{\widetilde{\varphi}^c}$ obtained from the appropriate bosonic action in Eq. (5) by subtracting the zeroth Matsubara components of Eq. (A2). The noise correlation function is given by

$$\langle\widetilde{\varphi}^c(\tau)\widetilde{\varphi}^c(\tau')\rangle = 2E_c\left[\delta(\tau-\tau')-\frac{1}{\beta}\right], \tag{C1}$$

which follows from the expansion of $\widetilde{\varphi}^c(\tau)$ in terms of Matsubara components: $\widetilde{\varphi}^c(\tau)=\sum_{m\neq 0}\widetilde{\varphi}^c_m \mathrm{e}^{-\mathrm{i}\omega_m\tau}$. Indeed, the functional distribution of $\widetilde{\varphi}^c_m$ is

$$\int \mathcal{D}[\widetilde{\varphi}^c(\tau)]\mathrm{e}^{-\int_0^\beta \mathrm{d}\tau\{\widetilde{\varphi}^c(\tau)[4E_c]^{-1}\widetilde{\varphi}^c(\tau)\}} = \int\prod_{m\neq 0} d\widetilde{\varphi}^c_m \mathrm{e}^{-\sum_{m,n\neq 0}\widetilde{\varphi}^c_m[\frac{\beta\delta_{m,-n}}{4E_c}]\widetilde{\varphi}^c_n}, \tag{C2}$$

which corresponds to $\langle\widetilde{\varphi}^c_m\widetilde{\varphi}^c_n\rangle = 2E_c\beta^{-1}\delta_{m,-n}$, immediately leading to Eq. (C1).

It is convenient to represent the noise field as $\widetilde{\varphi}^c(\tau)=\eta(\tau)+\mathrm{i}\zeta$ with $\eta(\tau)$ a random function and ζ a Gaussian random variable satisfying $\langle\widetilde{\eta}(\tau)\rangle=\langle\zeta\rangle=\langle\widetilde{\eta}(\tau)\zeta\rangle=0$, $\langle\widetilde{\eta}(\tau)\widetilde{\eta}(\tau')\rangle = 2E_c\delta(\tau-\tau')$, and $\langle\zeta^2\rangle = 2E_c/\beta$. As $\eta(\tau)$ is standard white noise, we follow the standard procedure[17] to map the Langevin equation (10a) to a Fokker-Planck (FP) equation:

$$\frac{\partial\mathcal{P}^c_\zeta}{\partial\tau} = \left(\frac{2\pi}{\beta}N-\mathrm{i}\zeta\right)\frac{\partial\mathcal{P}^c_\zeta}{\partial\theta}+E_c\frac{\partial^2\mathcal{P}^c_\zeta}{\partial\theta^2}. \tag{C3}$$

Here $\mathcal{P}^c_\zeta(\theta,\tau;\theta',\tau')$ is the *conditional* transition probability function for a given ζ, formally defined by $\mathcal{P}^c_\zeta(\theta,\tau;\theta',\tau') = \langle\delta[\theta(\tau)-\theta]\delta[\theta(\tau')-\theta']\rangle_\eta$, where the $\langle\cdots\rangle_\eta$ means averaging over the white noise $\eta(\tau)$. The full transition probability function $\mathcal{P}^c(\theta,\tau;\theta',\tau')$ is given by the subsequent averaging over the quenched (i.e., τ-independent) variable ζ (as, e.g., in Ref. 22):

$$\mathcal{P}^c(\theta,\tau;\theta',\tau') = \langle\delta[\theta(\tau)-\theta]\delta[\theta(\tau')-\theta']\rangle_{\eta,\zeta}, \tag{C4}$$

i.e., $\mathcal{P}^c \equiv \langle\mathcal{P}^c_\zeta\rangle_\zeta$.

Equation (C3) is a standard diffusion equation with a drift term. Its solution, with the natural boundary condition $\mathcal{P}^c(\theta,\tau;\theta',\tau|\zeta)=\delta(\theta-\theta')$, is a decaying Gaussian:

$$\mathcal{P}^c(\theta,\tau;\theta',\tau'|\zeta) = \frac{\exp\left\{-\frac{[(\theta-\theta')+(\frac{2\pi}{\beta}N-\mathrm{i}\zeta)|\tau-\tau'|]^2}{4E_c|\tau-\tau'|}\right\}}{\sqrt{4\pi E_c|\tau-\tau'|}}. \tag{C5}$$

Now we write $\Pi^c(\tau)$, defined in Eq. (A3), in terms of the transition probability function (C4):

$$\langle\mathrm{e}^{\mathrm{i}[\theta^c(\tau_f)-\theta^c(\tau_i)]}\rangle_{\widetilde{\varphi}^c} = \int_{-\infty}^{\infty} d\theta d\theta' \mathcal{P}^c(\theta,\tau_i;\theta',\tau_f)\mathrm{e}^{-\mathrm{i}(\theta-\theta')}.$$

Substituting here the solution (C5), we find the conditional (for a given ζ) phase-correlation function as

$$\Pi^c_\zeta(\tau) = \mathrm{e}^{-E_c\tau}\mathrm{e}^{-\zeta\tau}\mathrm{e}^{-\mathrm{i}2\pi N\frac{\tau}{\beta}}, \tag{C6}$$

where we defined $\tau=|\tau_f-\tau_i|$. Finally, the averaging over the quenched random variable ζ results in the first of Eqs. (A4). The second one, for $\Pi^s(\tau)$, was obtained by applying, step by step, exactly the same procedure.

APPENDIX D: CALCULATION OF THE CANONICAL PARTITION FUNCTION

In this Appendix, we evaluate the canonical partition functions Z_N and $Z_N(\not{q}_\alpha)$ defined in Eqs. (B1) and (B2). It follows from Eq. (B2) that

$$Z_N(\not{q}_\alpha) = \int_{-\pi}^{\pi}\frac{d\varphi^c}{2\pi}\mathrm{e}^{\mathrm{i}N\varphi^c}Z^0_\alpha(\mu=-\mathrm{i}\varphi^c/\beta) = \int_{-\pi}^{\pi}\frac{d\varphi^c}{2\pi}\mathrm{e}^{\mathrm{i}N\varphi^c}\prod_{\alpha'\neq\alpha}(1+\mathrm{e}^{-\beta\varepsilon_{\alpha'}-\mathrm{i}\varphi^c}). \tag{D1}$$

We calculate $Z_N(\not{q}_\alpha)$ (and thus Z_N) in the saddle-point approximation:

$$Z_N(\not{q}_\alpha) \approx \mathrm{e}^{-\mathcal{S}_\alpha(\varphi^c_0)}\int_{-\pi}^{\pi}\frac{d\varphi^c}{2\pi}\mathrm{e}^{-\frac{1}{2}[\mathcal{S}''_\alpha(\varphi^c_0)](\varphi^c-\varphi^c_0)^2}, \tag{D2}$$

where

$$\mathcal{S}_\alpha(\varphi^c) = -\mathrm{i}N\varphi^c - \ln\prod_{\alpha'\neq\alpha}(1+\mathrm{e}^{-\beta\varepsilon_{\alpha'}-\mathrm{i}\varphi^c}). \tag{D3}$$

The saddle-point equation, $\mathcal{S}'_\alpha=0$, is convenient to write by replacing $\sum_{\alpha'}f(\varepsilon_{\alpha'})$ with $\Delta^{-1}\int_0^\infty d\varepsilon f(\varepsilon)$ as in Appendix B. This gives, after calculating the integral, the following equation for finding φ^c_0:

$$N+\frac{1}{1+\mathrm{e}^{\beta\varepsilon_\alpha+\mathrm{i}\varphi^c_0}} = \frac{1}{\beta\Delta}\ln(1+\mathrm{e}^{-\mathrm{i}\varphi^c_0}). \tag{D4}$$

The Fermi factor there, being of order 1, can be neglected, which means that it is the same saddle point we would find in

a calculation of Z_N: for large enough N, the saddle point is unaltered by the exclusion of a single state. Assuming also that N is so large that $\beta N\Delta \gg 1$, we find from Eq. (D4)

$$-\mathrm{i}\varphi_0^c = \beta N\Delta \equiv \beta\bar{\mu}. \tag{D5}$$

In the same approximation, $S_\alpha(\varphi_0^c) = 1/(\beta\Delta)$, so that calculating the Gaussian integral in Eq. (D2) gives

$$Z_N = \mathrm{e}^{-\frac{1}{2}\beta\bar{\mu}N}, \tag{D6}$$

while $Z_N(\not{\alpha})$ differs only by the exclusion of the level α:

$$Z_N(\not{\alpha}) = [1 - n_\alpha(\bar{\mu})]\, Z_N. \tag{D7}$$

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